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States on Orthomodular Posets of Decompositions

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In Harding (*Transactions of American Mathematical Society* (1996) **348**(5), 1839–1862), it was shown that the direct product decompositions of a set X naturally form an orthomodular poset *Fact X*. Here it is shown that *Fact X* has a state if and only if X is finite. An example is also given of a finite orthomodular poset that can be embedded into *Fact X* for X countable, but not for X finite.

1. INTRODUCTION

In Harding (1996; see also Harding, 1998, 1999), a method was given to construct an orthomodular poset *Fact X* from the direct product decompositions of a set. As this forms the basis of our study we briefly review the pertinent facts. For a set X let *Eq X* denote the set of equivalence relations on X, use $\circ$ for relational product, Δ for the least equivalence relation on X, and ∇ for the largest. Define

$$\mathit{Fact}\,X = \{(\theta, \theta') \mid \theta, \theta' \in \mathit{Eq}\,X, \theta \cap \theta' = \Delta, \theta \circ \theta' = \nabla\}.$$

Let $\leq$ be the relation on *Fact X* defined by setting $(\theta, \theta') \leq (\phi, \phi')$ if $\theta \subseteq \phi, \phi' \subseteq \theta$, and all of $\theta, \theta', \phi, \phi'$ permute under relational product. Define a unary operation $\perp$ on *Fact X* by setting $(\theta, \theta')^{\perp} = (\theta', \theta)$. Then as shown in Harding (1996), $(\mathit{Fact}\ X, \leq, \perp)$ is an orthomodular poset with $(\theta, \theta') \vee (\phi, \phi') = (\theta \circ \phi, \theta' \cap \phi')$ for (θ, θ') orthogonal to (ϕ, ϕ'). The notation *Fact X* is used because such pairs (θ, θ') are commonly called factor pairs, and *Fact X* is called the orthomodular poset of decompositions of X because the factor pairs are exactly those pairs of equivalence relations that occur as the kernels of the projection operators associated with a binary direct product decomposition $X \cong Y \times Z$.

If X is the underlying set of some algebra A, then the factor pairs (θ, θ') which are compatible with this additional structure (i.e., which are congruences) form a suborthomodular poset *Fact A* of *Fact X*. For a vector space V the correspondence between subspaces of V and congruences of V, as well as the fact that all

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congruences of V permute, allows an easy description of *Fact V*. Letting $S(V)$ be the lattice of subspaces of V, up to isomorphism

$$\mathit{Fact}\,V = \{(S, S') \mid S, S' \in S(V), S \cap S' = \{0\}, S + S' = V\},$$

where $(S, S') \leq (T, T')$ iff $S \subseteq T$ and $T' \subseteq S'$, and $(S, S')^{\perp} = (S', S)$. Here, $(S, S') \vee (T, T') = (S + T, S' \cap T')$ when (S, S') is orthogonal to (T, T').

The reader is directed to Kalmbach (1983) and Pták and Pulmannová (1991) for general background on orthomodular lattices and orthomodular posets. The term suborthomodular poset means a subset S of an orthomodular poset P which is closed under orthocomplementation and finite orthogonal joins. A block of an orthomodular poset P is a maximal Boolean suborthomodular poset of P. A state on an orthomodular poset P is a map $\sigma : P \to [0, 1]$ such that (i) σ is order preserving, (ii) $\sigma(0) = 0$, (iii) $\sigma(x') = 1 - \sigma(x)$ for all $x \in P$, and (iv) $\sigma(x \vee y) = \sigma(x) + \sigma(y)$ for all $x \leq y'$.

2. A LEMMA

Lemma 2.1. *For V a three-dimensional vector space over the two element field Z_2, Fact V has exactly one state, and this state takes the value* $1/3$ *on each atom.*

Proof: The lattice of subspaces of any three-dimensional vector space is a projective plane where the one-dimensional subspaces are the points, the two-dimensional subspaces the lines, and incidence between points and lines is given by set containment. For V a three-dimensional vector space over Z_2, the associated seven-point projective plane is commonly called the Fano plane. We use the geometric notation of $\overline{pq}$ for the line determined by two distinct points p, q and $l \wedge m$ for the unique intersection point of the distinct lines l, m. So the atoms of *Fact V* are the ordered pairs (p, l) where p is a point, l is a line, with $p \not{I}\; l$. Note that $(p_0, l_0) \perp (p_1, l_1)$ iff $p_0 \; I \; l_1$ and $p_1 \; I \; l_0$. The following claim is immediate.

Claim 1. Each block of *Fact V* has exactly three atoms, and the third atom of the block containing the atoms (p, l) and (q, m) is $(l \wedge m, \overline{pq})$.

Suppose σ is a state on *Fact V*.

Claim 2. $\sigma(p, l) = \sigma(q, l)$ for any line l and any points, $p, q \not{I}\; l$.

Proof: Let $\overline{pq} = m$ and $l \wedge m = r$. Suppose s, t are the other two points on the line l. Then $(p, l), (q, l) \perp (s, m), (t, m)$. Hence, by Claim 1, we have the following blocks:

$$(p, l), (s, m), (r, \overline{ps}) \tag{2.1}$$

$$(p, l), (t, m), (r, \overline{pt}) \tag{2.2}$$

$$(q, l), (s, m), (r, \overline{qs}) \tag{2.3}$$

$$(q, l), (t, m), (r, \overline{qt}) \tag{2.4}$$

Let n be the third line through the point r, hence n contains none of the points p, q, s, t. Let u, v be the points on n that differ from r. As $\overline{sp}$ intersects n at a point different from r, either *spu* are collinear, or *spv* are collinear. We may assume *spu* are collinear. It follows that *tpv*, *sqv*, and *tqu* are all sets of collinear points. Then, by Claim 1, we have the following blocks:

$$(u, l), (s, n), (r, \overline{ps}) \tag{2.5}$$

$$(u, l), (t, n), (r, \overline{qt}) \tag{2.6}$$

$$(v, l), (s, n), (r, \overline{qs}) \tag{2.7}$$

$$(v, l), (t, n), (r, \overline{pt}) \tag{2.8}$$

As the values of the state σ on the atoms of a block sum to 1 we can therefore compare (2.1) and (2.5), (2.4) and (2.6), (2.3) and (2.7), (2.2) and (2.8) to obtain

$$\sigma(u, l) + \sigma(s, u) = \sigma(p, l) + \sigma(s, m) \tag{2.9}$$

$$\sigma(u, l) + \sigma(t, n) = \sigma(q, l) + \sigma(t, m) \tag{2.10}$$

$$\sigma(v, l) + \sigma(s, n) = \sigma(q, l) + \sigma(s, m) \tag{2.11}$$

$$\sigma(v, l) + \sigma(t, n) = \sigma(p, l) + \sigma(t, m) \tag{2.12}$$

Comparing (2.9) minus (2.10) and (2.11) minus (2.12) we obtain

$$\sigma(p, l) + \sigma(s, m) - \sigma(q, l) - \sigma(t, m) = \sigma(q, l) + \sigma(s, m) - \sigma(p, l) - \sigma(t, m).$$

Hence $2\sigma(p, l) = 2\sigma(q, l)$, establishing our claim.

Claim 3. $\sigma(p, l) = \sigma(p, m)$ for any point p and lines l, m with $p \not{I}\ l, m$.

Proof: Let $r = l \wedge m$ and n be a line with $p\ I\ n$ and $r \not{I}\ n$. We have the blocks

$$(r, n), (p, l), (l \wedge n, \overline{rp})$$

$$(r, n), (p, m), (m \wedge n, \overline{rp})$$

By the previous claim $\sigma(l \wedge n, \overline{rp}) = \sigma(m \wedge n, \overline{rp})$ hence $\sigma(p, l) = \sigma(p, m)$.

We are now in a position to prove the lemma. Let (p, l), (q, m) be any two atoms of *Fact V*. There is a point $r \not\prec l, m$. Then by Claims 2 and 3, $\sigma(p, l) = \sigma(r, l) = \sigma(r, m) = \sigma(q, m)$. As each block has exactly three atoms, it follows that $\sigma(p, l) = 1/3$ for each atom (p, l); hence, *Fact V* has at most one state. But there is a state on *Fact V* taking the value $1/3$ on each atom, namely, the state given by $\sigma(S, S') = \dim(S)/3$ (see Harding, 1996, for details). □

3. APPLICATIONS OF THE LEMMA

We will require some notation for standard notations about vector spaces. Let κ be a cardinal. Recall that a basis of a κ-dimensional vector space W is a map $\xi : \kappa \to W$ with $\xi_\alpha \neq \xi_\beta$ for $\alpha \neq \beta$ such that the image of ξ is an independent set which spans W. Given such a basis ξ, for each element $w \in W$ one has a unique family of scalars w_α, nonzero for only finitely many α, such that $w = \sum w_\alpha \xi_\alpha$. Henceforth, we assume V is a three-dimensional vector space over the two-element field Z_2 and that the sequence $\zeta_0, \zeta_1, \zeta_2$ is a basis for V. As before we use $S(V)$ for the lattice of subspaces of V. Frequent use will be made of the fact that each ordinal β has a unique representation $\beta = \alpha + n$ where α is a limit ordinal and n is a natural number. We use ω for the set of natural numbers.

Lemma 3.1. *Let κ be an infinite cardinal, W be a κ-dimensional vector space over Z_2, and $\xi : \kappa \to W$ be a basis of W. Then $\psi_\xi : S(V) \to S(W)$ defined by*

$$\psi_\xi(S) = \{w \in W \mid w_{\alpha+3n}\zeta_0 + w_{\alpha+3n+1}\zeta_1 + w_{\alpha+3n+2}\zeta_2 \in S \text{ for each } n \in \omega \text{ and limit ordinal } \alpha < \kappa\}$$

is a bounded lattice embedding.

Proof: As $(\lambda w + \lambda' w')_\epsilon = \lambda w_\epsilon + \lambda' w'_\epsilon$, it follows easily that $\psi_\xi(S)$ is indeed a subspace of W. It is clear that the map ψ_ξ is order preserving, hence $\psi_\xi(S \cap T) \subseteq \psi_\xi(S) \cap \psi_\xi(T)$ and $\psi_\xi(S) + \psi_\xi(T) \subseteq \psi_\xi(S + T)$. To show that ψ_ξ is a lattice homomorphism, it remains to show the reverse inequalities. Suppose $w \in \psi_\xi(S) \cap \psi_\xi(T)$. For each limit ordinal $\alpha < \kappa$ and each natural number n, we have $w_{3n}\zeta_0 + w_{3n+1}\zeta_1 + w_{3n+2}\zeta_2$ in both S and T, hence in $S \cap T$. Thus $\psi_\xi(S) \cap \psi_\xi(T) \subseteq \psi_\xi(S \cap T)$.

Claim. For any subspace R of V, the set $\{a\xi_{\alpha+3n} + b\xi_{\alpha+3n+1} + c\xi_{\alpha+3n+2} \mid \alpha < \kappa$ is a limit ordinal, $n \in \omega$ and $a\zeta_0 + b\zeta_1 + c\zeta_2 \in R\}$ spans $\psi_\xi(R)$.

Proof: If $w \in \psi_\xi(R)$ then $w_{\alpha+3n}\xi_{\alpha+3n} + w_{\alpha+3n+1}\xi_{\alpha+3n+1} + \omega_{\alpha+3n+2}\xi_{\alpha+3n+2}$ also belongs to $\psi_\xi(R)$ and clearly w is a finite sum of such vectors.

To show $\psi_\xi(S+T) \subseteq \psi_\xi(S) + \psi_\xi(T)$, it suffices to provide a spanning set of $\psi_\xi(S+T)$ contained in $\psi_\xi(S) + \psi_\xi(T)$. Let $\alpha < \kappa$ be a limit ordinal, n be a natural number, and suppose $w = a\xi_{\alpha+3n} + b\xi_{\alpha+3n+1} + c\xi_{\alpha+3n+2}$ where $a\zeta_0 + b\zeta_1 + c\zeta_2$ is in $S + T$. Then there are $a'\zeta_0 + b'\zeta_1 + c'\zeta_2 \in S$ and $a''\zeta_0 + b''\zeta_1 + c''\zeta_2 \in T$ with

$$a\zeta_0 + b\zeta_1 + c\zeta_2 = (a'\zeta_0 + b'\zeta_1 + c'\zeta_2) + (a''\zeta_0 + b''\zeta_1 + c''\zeta_2).$$

Setting $w' = a'\xi_{\alpha+3n} + b'\xi_{\alpha+3n+1} + c'\xi_{\alpha+3n+2}$ and $w'' = a''\xi_{\alpha+3n} + b''\xi_{\alpha+3n+1} + c''\xi_{\alpha+3n+2}$ we then have $w = w' + w''$ and clearly $w' \in \psi_\xi(S)$ and $w'' \in \psi_\xi(T)$. Thus $w \in \psi_\xi(S) + \psi_\xi(T)$, showing that $\psi_\xi(S+T) \subseteq \psi_\xi(S) + \psi_\xi(T)$.

Having established that ψ_ξ is a lattice homomorphism, note that ψ_ξ clearly preserves bounds. As $S(V)$ is a complemented modular lattice, each congruence of $S(V)$ is determined by its zero equivalence class. But $\psi_\xi(S) = \{0\}$ iff $S = \{0\}$. Therefore ψ_ξ is an embedding. □

In the following lemma we assume κ is an infinite cardinal, W is a κ-dimensional vector space over Z_2, and $\xi : \kappa \to W$ is a basis of W.

Lemma 3.2. *The map $\psi_\xi^{(2)}$: Fact V → Fact W defined by $\psi_\xi^{(2)}(S, S') = (\psi_\xi(S), \psi_\xi(S'))$ is an order embedding which preserves orthocomplementation and finite orthogonal joins. Hence Fact V is isomorphic to a suborthomodular poset of Fact W.*

Proof: Suppose (S, S') and (T, T') are elements of *Fact V*. As $(S, S') \leq (T, T')$ iff $S \leq T$ and $T' \leq S'$, which occurs iff $\psi_\xi(S) \leq \psi_\xi(T)$ and $\psi_\xi(T') \leq \psi_\xi(S')$, we have that $\psi_\xi^{(2)}$ is an order embedding. As $(S, S')^\perp = (S', S)$ one easily sees that $\psi_\xi^{(2)}$ preserves orthocomplementation. If (S, S') and (T, T') are orthogonal, then $(S, S') \vee (T, T') = (S+T, S' \cap T')$. So, if (S, S') and (T, T') are orthogonal in *Fact V*, then as $\psi_\xi^{(2)}$ is order and orthocomplement preserving we have $\psi_\xi^{(2)}(S, S')$ and $\psi_\xi^{(2)}(T, T')$ orthogonal in *Fact W* and $\psi_\xi^{(2)}((S, S') \vee (T, T')) = \psi_\xi^{(2)}(S+T, S' \cap T') = (\psi_\xi(S+T), \psi_\xi(S' \cap T')) = (\psi_\xi(S) + \psi_\xi(T), \psi_\xi(S') \cap \psi_\xi(T')) = \psi_\xi^{(2)}(S, S') \vee \psi_\xi^{(2)}(T, T')$. □

Theorem 3.3. *For W a vector space over Z_2, Fact W has a state iff W is finite dimensional.*

Proof: If W is of finite dimension n, define σ : *Fact V* $\to [0, 1]$ by setting $\sigma(S, S') = \dim(S)/n$. Clearly σ is order preserving and $\sigma(\{0\}, W) = 0$. If S, T are subspaces with trivial intersection, then $\dim(S+T) = \dim(S) + \dim(T)$. It follows that $\sigma((S, S')^\perp) = \dim(S')/n = 1 - \dim(S)/n = 1 - \sigma(S, S')$. And if

(S, S') is orthogonal to (T, T') then as $S \cap T = \{0\}$ we have $\sigma((S, S') \vee (T, T')) = \sigma(S + T, S' \cap T') = \dim(S + T)/n = \dim(S)/n + \dim(T)/n = \sigma(S, S') + \sigma(T, T')$.

Suppose W is of infinite dimension κ and let V be a three-dimensional vector space over Z_2 with basis $\zeta_0, \zeta_1, \zeta_2$.

Claim. If A, A' are complementary subspaces of W, each of dimension κ, then there is a basis ξ of W such that (A, A') is the image under $\psi_\xi^{(2)} : \mathit{Fact}\, V \to \mathit{Fact}\, W$ of an atom of *Fact V*.

Proof: Let $\mu : \kappa \to A$ and $\nu : \kappa \to A'$ be bases. Recalling that each element of κ can be uniquely expressed as $\alpha + n$ for some limit ordinal α and some natural number n define $\xi : \kappa \to W$ by setting $\xi_{\alpha+3n} = \mu_{\alpha+n}, \xi_{\alpha+3n+1} = \nu_{\alpha+2n}$ and $\xi_{\alpha+3n+2} = \nu_{\alpha+2n+1}$. Then for S the subspace of V spanned by ζ_0, and S' the subspace of V spanned by ζ_1, ζ_2, we have $\psi_\xi(S) = A$ and $\psi_\xi(S') = A'$. It follows that (A, A') is the image under $\psi_\xi^{(2)}$ of the atom (S, S') of *Fact V*.

To conclude the proof, suppose $\sigma : \mathit{Fact}\, W \to [0, 1]$ is a state. Choose A, A' complementary κ-dimensional subspaces of W. Then there is a basis ξ such that (A, A') is the image under $\psi_\xi^{(2)}$ of an atom of *Fact V*, and as $\sigma \circ \psi_\xi^{(2)}$ is a state on *Fact V* it follows from Lemma 2.1 that $\sigma(A, A') = 1/3$. But we may also apply this process to obtain a basis ξ' such that (A', A) is the image under $\psi_{\xi'}^{(2)}$ of an atom of *Fact V*, hence $\sigma(A', A) = 1/3$. This contradiction shows *Fact W* has no state for W infinite dimensional. □

Theorem 3.4. *For a set X, Fact X has a state iff X is finite and has more than one element.*

Proof: Suppose first that X is an infinite set of cardinality κ. As there is a vector space of cardinality κ over Z_2, there is also a vector space W over Z_2 having underlying set X. Clearly *Fact W* is a suborthomodular poset of *Fact X*, so it follows from the previous theorem that *Fact X* has no state.

Suppose now that X is a finite set with more than one element and let $m = \ln |X|$. Define a map $\sigma : \mathit{Fact}\, X \to [0, 1]$ by setting $\sigma(\theta, \theta') = (\ln |X/\theta'|)/m$. Then we have $\sigma(\Delta, \nabla) = (\ln 1)/m = 0$ and σ is clearly order preserving. For (θ, θ') a factor pair, $X \cong X/\theta \times X\theta'$ so $\sigma(\theta, \theta') + \sigma(\theta', \theta) = (\ln |X/\theta'| + \ln |X/\theta|)/m$, which by basic properties of logarithms equals $(\ln(|X/\theta'| \times |X/\theta|)/m = (\ln |X|)/m$. It follows that $\sigma((\theta, \theta')^\perp) = 1 - \sigma(\theta, \theta')$. If (θ, θ') and (ϕ, ϕ') are orthogonal, then as shown in Harding (1996), $X \cong X/\theta' \times X/(\theta \circ \phi) \times X/\phi'$. It follows that $\sigma(\theta, \theta') + \sigma(\theta' \cap \phi', \theta \circ \phi) + \sigma(\phi, \phi') = 1$ and therefore that $\sigma((\theta, \theta') \vee (\phi, \phi')) = \sigma(\theta, \theta') + (\phi, \phi')$. Thus for X finite and having more than one element *Fact X* has a state. If X has at most one element, then *Fact X* is the one element orthomodular poset which has no state. □

Theorem 3.5. *There is a finite orthomodular poset P which is isomorphic to a suborthomodular poset of Fact X for X countable, but not isomorphic to a suborthomodular poset of Fact X for any finite set X.*

Proof: Let V be a three-dimensional vector space over Z_2 with basis $\zeta_0, \zeta_1, \zeta_2$. Let S be the subspace of V spanned by ζ_0, and S' be the subspace of V spanned by ζ_1, ζ_2. Then (S, S') is an atom of *Fact V*. Let P be the orthomodular poset consisting of four copies of *Fact V* with an extra block having as its atoms the four copies of (S, S'). Surely P is finite. By Lemma 2.1 any state on *Fact V* takes value $1/3$ on each atom, hence there is no state on P as the sum of the values of the state on the atoms of the extra block would be $4/3$. By the previous theorem, P is not isomorphic to a suborthomodular poset of *Fact X* for any finite set X.

To show that P can be embedded into *Fact X* for X countable, it is enough to show P is isomorphic to a suborthomodular poset of *Fact W* for W a countable dimension vector space over Z_2, say with basis $\xi : \omega \to W$.

Claim 1. There exist bases $\xi^i : \omega \to W$ for $i = 0, 1, 2, 3$ so that for each $i \neq j$

(1) $\exists n, m\big(\xi^i_{3n} = \xi^j_{3m+1}$ and $\{\xi^i_{3n}, \xi^i_{3n+1}, \xi^i_{3n+2}\} \cap \{\xi^j_{3m}, \xi^j_{3m+1}, \xi^j_{3m+2}\} = \xi^i_{3n}\big)$

(2) $\exists n, m\big(\xi^i_{3n} = \xi^j_{3m+2}$ and $\{\xi^i_{3n}, \xi^i_{3n+1}, \xi^i_{3n+2}\} \cap \{\xi^j_{3m}, \xi^j_{3m+1}, \xi^j_{3m+2}\} = \xi^i_{3n}\big)$

(3) $\exists n, m\big(\xi^i_{3n+1} = \xi^j_{3m}$ and $\{\xi^i_{3n}, \xi^i_{3n+1}, \xi^i_{3n+2}\} \cap \{\xi^j_{3m}, \xi^j_{3m+1}, \xi^j_{3m+2}\} = \xi^i_{3n+1}\big)$

(4) $\exists n, m\big(\xi^i_{3n+1} = \xi^j_{3m+1}$ and $\{\xi^i_{3n}, \xi^i_{3n+1}, \xi^i_{3n+2}\} \cap \{\xi^j_{3m}, \xi^j_{3m+1}, \xi^j_{3m+2}\} = \xi^i_{3n+1}\big)$

(5) $\exists n, m\big(\xi^i_{3n+1} = \xi^j_{3m+2}$ and $\{\xi^i_{3n}, \xi^i_{3n+1}, \xi^i_{3n+2}\} \cap \{\xi^j_{3m}, \xi^j_{3m+1}, \xi^j_{3m+2}\} = \xi^i_{3n+1}\big)$

(6) $\exists n, m\big(\xi^i_{3n+2} = \xi^j_{3m}$ and $\{\xi^i_{3n}, \xi^i_{3n+1}, \xi^i_{3n+2}\} \cap \{\xi^j_{3m}, \xi^j_{3m+1}, \xi^j_{3m+2}\} = \xi^i_{3n+2}\big)$

(7) $\exists n, m\big(\xi^i_{3n+2} = \xi^j_{3m+1}$ and $\{\xi^i_{3n}, \xi^i_{3n+1}, \xi^i_{3n+2}\} \cap \{\xi^j_{3m}, \xi^j_{3m+1}, \xi^j_{3m+2}\} = \xi^i_{3n+2}\big)$

(8) $\exists n, m\big(\xi^i_{3n+2} = \xi^j_{3m+2}$ and $\{\xi^i_{3n}, \xi^i_{3n+1}, \xi^i_{3n+2}\} \cap \{\xi^j_{3m}, \xi^j_{3m+1}, \xi^j_{3m+2}\} = \xi^i_{3n+2}\big)$

(9) $\{\xi^i_{3n} \mid n \in \omega\} = \{\xi_{4n+i} \mid n \in \omega\}$.

Proof: The bases ξ^i can be constructed by simply rearranging the sequencing of the basis ξ. To construct ξ^i we first fill every third spot of ξ^i with a basis element of the form ξ_{4n+i}. This can be done simply by setting $\xi^i_{3n} = \xi_{4n+i}$. This ensures the final condition. Note that the other eight conditions are of a finitary nature—each refers to one triple of elements from each of the bases ξ^i, ξ^j. Also, as these conditions are to be read as logical statements, the choice of triples may vary from condition to condition, that is, a different *m, n* may be chosen for each condition. One quickly sees it is trivial to fill in triples of ξ^i, ξ^j to ensure any one of the conditions, and by separating the triples sufficiently, all eight conditions may be established for all $i \neq j$. We then have assigned elements to every third spot of each sequence ξ^i and to finitely many other spots. One then simply fills in the remaining spots of ξ^i in a manner which exhausts the members of ξ which have not already been used in ξ^i.

For (S, S') the atom of *Fact V* described above we have the following.

Claim 2. If A and B are subspaces of V distinct from the bounds and $i \neq j$, then $\psi_{\xi^i}(A) \subseteq \psi_{\xi^j}(B)$ iff $A = S$ and $B = S'$.

Proof: Assume $\psi_{\xi^i}(A) \subseteq \psi_{\xi^j}(B)$ and that $a\zeta_0 + b\zeta_1 + c\zeta_2 \in A$. By definition of ψ_{ξ^i} it follows that

$$w^n = a\xi^i_{3n} + b\xi^i_{3n+1} + c\xi^i_{3n+2}$$

belongs to $\psi_{\xi^i}(A)$ and hence to $\psi_{\xi^j}(B)$ for any $n \in \omega$. Consider the expansion of w^n with respect of the basis ξ^j. Choose $m \in \omega$, let a' be the coefficient of the ξ^j_{3m} term of this expansion, b' be the coefficient of the ξ^j_{3m+1} term, and c' be the coefficient of the ξ^j_{3m+2} term. Set

$$w^{n,m} = a'\zeta_0 + b'\zeta_1 + c'\zeta_2.$$

As w^n belongs to $\psi_{\xi^j}(B)$ for each $n \in \omega$, it follows that $w^{n,m}$ belongs to B for each $m, n \in \omega$. Applying this observation in conjunction with Conditions (1) and (2) we have $a\zeta_1, a\zeta_2 \in B$. Similarly Conditions (3)–(5) provide $b\zeta_0, b\zeta_1, b\zeta_2 \in B$ and Conditions (6)–(8) provide $c\zeta_0, c\zeta_1, c\zeta_2 \in B$. As B is not the whole of V it follows that $b = c = 0$. As A is not trivial, it follows that $a \neq 0$. This shows both $A = S$ and $B = S'$.

Conversely, the elements of $\psi_{\xi^i}(S)$ are precisely those that have zeros in all but the $3n$ spots of their ξ^i expansion. Thus, by Condition (9), $\psi_{\xi^i}(S)$ is the subspace spanned by $\{\xi_{4n+i} \mid n \in \omega\}$. But the elements of $\psi_{\xi^j}(S')$ are exactly the ones which have zeros in the $3n$ spots of their ξ^j expansions. Hence, by Condition (9), $\psi_{\xi^j}(S')$ is the subspace spanned by $\{\xi_n \mid n \in \omega\} - \{\xi_{4n+j} \mid n \in \omega\}$. So $\psi_{\xi^i}(S) \subseteq \psi_{\xi^j}(S')$ concluding the proof of the claim.

To conclude the proof of the theorem, let Q be the subset of *Fact W* consisting of the union of the images of the maps $\psi^{(2)}_{\xi^i} : \mathit{Fact}\, V \to \mathit{Fact}\, W$ for $i = 0, 1, 2, 3$. From Claim 2 the images of the atom (S, S') under these maps are pairwise orthogonal and these are the only orthogonalities between members of the images of different maps. Clearly Q is closed under orthocomplementation, and Q is nearly closed under finite orthogonal joins as well. It is only the images of (S, S') which are missing orthogonal joins. Let B be the 16-element Boolean suborthomodular poset of *Fact W* generated by the images of (S, S') and set $Q' = Q \cup B$. Note, if x belongs to the image of $\psi_{\xi^i}^{(2)}$, and y, z are distinct atoms of B, then x is not orthogonal to $y \vee z$ as this would imply x is orthogonal to both y, z contrary to Claim 2. Therefore there are no new orthogonalities in Q' which are not already present in either Q or B. It follows that Q' is closed under finite orthogonal joins and that B intersects Q only at the bounds and the four atoms. It follows that Q' is a suborthomodular poset of *Fact W* that is isomorphic to P. □

4. CONCLUSIONS

There are examples of finite orthomodular posets that cannot be embedded into *Fact X* for any set X (Harding, 1996, contains one such example, and its method can be used to create many more). However, one can show that that the known examples of orthomodular posets which cannot be embedded into *Fact X* also cannot be embedded into any orthomodular lattice. A question occurs whether every orthomodular lattice can be embedded into *Fact X* by an embedding which preserves orthocomplementation and finite orthogonal joins. A positive answer to this question would be most appealing. It would have the results of this paper as trivial consequences (there is a finite orthomodular lattice without any state) and moreover would provide an analog of Cayley's theorem for orthomodular lattices. The methods presented here seem insufficient to tackle this problem. However, one at least obtains the useful information that to embed an orthomodular lattice L into *Fact X*, the set X one constructs must be infinite even if L is finite.

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